ELSEVIER

Journal of Pure and Applied Algebra 153 (2000) 1–15

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Mori domains of integer-valued polynomials

Paul-Jean Cahen[a,1], Stefania Gabelli[b,2], Evan Houston[c,*,3]

[a] *Service de Mathématiques 322, Faculté des Sciences de Saint-Jérôme, 13397 Marseille cedex 20, France*

[b] *Dipartimento di Matematica, Università degli Studi Roma Tre, Largo San L. Murialdo, 1, 00146 Roma, Italy*

[c] *Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA*

Received 23 November 1998
Communicated by A.V. Geramita

Abstract

Let D be a domain with quotient field K. We investigate conditions under which the ring $\mathrm{Int}(D)=\{f\in K[X]\,|\,f(D)\subseteq D\}$ of integer-valued polynomials over D is a Mori domain. In particular, we show that if D is a pseudo-valuation domain with finite residue field such that the associated valuation overring is rank one discrete and has infinite residue field, then $\mathrm{Int}(D)$ is a Mori domain with $\mathrm{Int}(D)\neq D[X]$. Finally, we investigate the class group of a Mori domain of integer-valued polynomials, showing, in the case just mentioned, that $\mathrm{Cl}(\mathrm{Int}(D))$ is generated by the classes of the t-maximal uppers to zero. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 13G05; 13C20; 13F20

1. Introduction

Let D be a domain with quotient field K. We denote by $\mathrm{Int}(D)$ the ring of integer-valued polynomials on D: $\mathrm{Int}(D)=\{f\in K[X]\,|\,f(D)\subseteq D\}$. Rings of this type have received a great deal of attention in recent years, in part because they often provide

* Corresponding author.
E-mail addresses: paul-jean.cahen@math.u-3mrs.fr (P. Cahen), gabelli@mat.uniroma3.it (S.J. Gabelli), eghoust@email.uncc.edu (E. Houston)

[1] The first author acknowledges support from C.N.R.S. UMR 6632
[2] The second author acknowledges support from Ministero dell'Università e della Ricerca Scientifica e Tecnologica d'Italia and from NATO Grant 970140.
[3] The third author acknowledges support from the Consiglio Nazionale delle Ricerche d'Italia, from C.N.R.S. UMR 6632, and from NATO Grant 970140.

0022-4049/00/$ - see front matter
PII: S0022-4049(99)00073-0

interesting examples in various classical settings. For instance, if V is a complete rank one discrete valuation domain with finite residue field, then $\mathrm{Int}(V)$ is a completely integrally closed two-dimensional Prüfer domain which is not the intersection of rank one valuation domains and which has localizations which are not completely integrally closed (see [11, pp. 130–131]).

In this paper we study the Mori property in $\mathrm{Int}(D)$. Recall that D is a Mori domain if it satisfies the ascending chain condition on divisorial ideals. The class of Mori domains includes Noetherian domains and Krull domains, and considerable research has been devoted to trying to prove that various properties held by Noetherian and/or Krull domains are also held by Mori domains.

We begin by showing that, if D is a Mori domain and S is a multiplicative subset of D, then $(\mathrm{Int}(D))_S = \mathrm{Int}(D_S)$ (generalizing the result already known for Noetherian D). We also show that in order that $\mathrm{Int}(D)$ be a Mori domain different from $D[X]$, it is necessary that D be a Mori domain with D/Q infinite for each v-invertible maximal divisorial ideal Q of D and that D/P be finite for some (non-v-invertible) maximal divisorial ideal P of D.

In Section 2, we construct examples of the type just mentioned, that is, domains D such that $\mathrm{Int}(D)$ is a Mori domain with $\mathrm{Int}(D) \neq D[X]$. Indeed, we show that D can be taken to be a local domain with maximal ideal I such that D/I is finite and such that I is also a nonzero proper principal ideal of a local Noetherian overring B having infinite residue field. (For example, let k be a finite field, and let L be an infinite field containing k; then set $B = L[[t]]$, $I = aB$, where a is a nonzero nonunit of B, and $D = k + I$.) The existence of these examples for one-dimensional D is especially interesting in light of a result of Gilmer et al. In [16] they show that if D is a one-dimensional *Noetherian* domain with $\mathrm{Int}(D) \neq D[X]$, then $\mathrm{Int}(D)$ cannot be Noetherian. We generalize their result by showing that $\mathrm{Int}(D)$ cannot even be Mori.

In Section 3, we study class groups. Recall that the (t)-class group of a domain D is defined by $\mathrm{Cl}(D)=\mathscr{T}(D)/\mathscr{P}(D)$, where $\mathscr{T}(D)$ is the group of t-invertible t-ideals of D, and $\mathscr{P}(D)$ is the subgroup of principal ideals. We show that the extension $D \subseteq \mathrm{Int}(D)$ is compatible for the t-operation, that is, that $J_t \subseteq (J\,\mathrm{Int}(D))_t$ for each nonzero ideal J of D. It follows that the map $\mathrm{Cl}(D) \to \mathrm{Cl}(\mathrm{Int}(D))$, given by $[J] \mapsto [(J\,\mathrm{Int}(D))_t]$, is a homomorphism. In fact, we show that the following sequence is exact:

$$0 \longrightarrow \mathrm{Cl}(D) \xrightarrow{i} \mathrm{Cl}(\mathrm{Int}(D)) \xrightarrow{p} \prod_{M \in t\text{-}\max(D)} \mathrm{Cl}(\mathrm{Int}(D_M)).$$

(Here, t-max(D) denotes the set of maximal t-ideals of D.) We then show that, in the case where D is a Mori domain in which every maximal t-ideal has height one, the following sequence is exact:

$$0 \longrightarrow \mathrm{Cl}(D) \xrightarrow{i} \mathrm{Cl}(\mathrm{Int}(D)) \xrightarrow{p} \bigoplus_{M \in t\text{-}\max(D)} \mathrm{Cl}(\mathrm{Int}(D_M)) \longrightarrow 0.$$

Finally, we observe that, in the construction of Section 2, we have that the class group of Int (D) is generated by the classes of the uppers to zero which are maximal t-ideals.

2. When is Int(D) Mori?

In this section, we generalize to Mori domains some results on integer-valued polynomials which are known for Noetherian and/or Krull domains. We begin by reviewing some relevant definitions. First, recall that D denotes a domain with quotient field K. For a fractional ideal I of D, we denote by I^{-1} the set $\{x \in K \mid xI \subseteq D\}$. The v-closure I_v of I is then given by $I_v = (I^{-1})^{-1}$ (which is equal to the intersection of the principal fractional ideals which contain I), and the t-closure is given by $I_t = \cup\{J_v \mid J$ is a finitely generated ideal of D with $J \subseteq I\}$. An ideal I is divisorial (respectively, a t-ideal) if $I = I_v$ (respectively, $I = I_t$). The domain D is a Mori domain if it satisfies the ascending chain condition on divisorial ideals, equivalently, if for each ideal I of D there is a finitely generated ideal J of D with $J \subseteq I$ and $J_v = I_v$ [21, Théorème 1].

Proposition 2.1 (cf. [11, Proposition I.2.3]). *If D is a Mori domain and S is a multiplicative subset of D, then* $(\mathrm{Int}(D))_S = \mathrm{Int}(D_S)$.

Proof. We have $(\mathrm{Int}(D))_S \subseteq \mathrm{Int}(D_S)$ (for an arbitrary domain D) by [11, Proposition I.2.2]. Let $f \in \mathrm{Int}(D_S)$, and consider the D-submodule I of K generated by $f(D)$. We have $I \subseteq D_S \cap c(f)$, where $c(f)$ is the *content* of f, that is, the fractional D-ideal generated by the coefficients of f. Thus I is a fractional ideal of D which is contained in D_S. Since D is a Mori domain, there is a finitely generated ideal $J \subseteq I$ with $I_v = J_v$, and since $I \subseteq D_S$, there is an element $s \in S$ with $sJ \subseteq D$. It follows that $sf(D) \subseteq sJ_v \subseteq D$. Therefore, $f \in (\mathrm{Int}(D))_S$, as desired. □

The proof of Proposition 2.1 just required a standard “Mori variation” of the proof given in the Noetherian case in [11, Theorem I.2.3]. In a similar way, we can obtain an analogue of Cahen and Chabert [11, Proposition I.2.7]. Recall that a subset E of K is called a fractional subset of D if $dE \subseteq D$ for some nonzero element $d \in D$. We then denote by $\mathrm{Int}(E,D)$ the ring of integer-valued polynomials on E, that is, $\mathrm{Int}(E,D) = \{f \in K[X] \mid f(E) \subseteq D\}$.

Proposition 2.2. (1) *Let D be a Mori domain, E a fractional subset of D, and S a multiplicative subset of D. Then* $(\mathrm{Int}(E,D))_S = \mathrm{Int}(E,D_S)$.

(2) *Let R be a Mori subring of a domain D and S a multiplicative subset of R. Then* $(\mathrm{Int}(R,D))_S = \mathrm{Int}(R,D_S) = \mathrm{Int}(R_S,D_S)$.

In particular, as an immediate corollary, we obtain the following generalization of Cahen and Chabert [11, Proposition I.2.8].

Corollary 2.3. *If D is a Krull domain, S is a multiplicative subset of D, and E is a fractional subset of D, then* $(\mathrm{Int}(E,D))_S = \mathrm{Int}(E,D_S)$.

Our next result characterizes when the ring of integer-valued polynomials over a Mori domain is trivial (that is, is equal to the polynomial ring over the domain).

Proposition 2.4. *Let D be a Mori domain. Then* $\mathrm{Int}(D)=D[X]$ *if and only if each maximal divisorial ideal of D has infinite residue field.*

Proof. First, suppose that each maximal divisorial ideal D has infinite residue field. Then, since $D=\cap\{D_P \mid P$ is a maximal divisorial ideal of $D\}$ [6, Proposition 2.2], we have that $\mathrm{Int}(D)=D[X]$ by [11, Corollary I.3.6]. Conversely, suppose that $\mathrm{Int}(D)=D[X]$, and let P be a maximal divisorial ideal of D. Then by [17, Proposition 2.3], P is a conductor ideal, and [11, Proposition I.3.10] then implies that D/P is infinite. □

In Krull domains and one-dimensional Mori domains, the maximal divisorial ideals are the height one primes. Hence we have the following generalization of Cahen and Chabert [11, Corollary I.3.15].

Corollary 2.5. *Let D be a Krull domain or a one-dimensional Mori domain. Then* $\mathrm{Int}(D)=D[X]$ *if and only if the residue field of each height-one prime ideal of D is infinite.*

We now give two necessary conditions for $\mathrm{Int}(D)$ to be Mori.

Proposition 2.6. *If* $\mathrm{Int}(D)$ *is Mori, then*
(1) *D is Mori, and*
(2) *each v-invertible maximal divisorial ideal of D has infinite residue field.*

Proof. Assume that D is a domain such that $\mathrm{Int}(D)$ is Mori. That D is Mori then follows from the fact that $D=\mathrm{Int}(D)\cap K$ and [23, I, Théorème 2]. To prove (2), suppose by way of contradiction that P is a v-invertible maximal divisorial ideal of D with finite residue field. Then D_P is a rank one discrete valuation domain [6, Theorem 2.5] with finite residue field. It follows that $\mathrm{Int}(D_P)$ is two-dimensional [11, Proposition V.1.8] and Prüfer [11, Lemma VI.1.4]. However, by Lemma 2.1, $\mathrm{Int}(D_P)$ is a ring of fractions of $\mathrm{Int}(D)$ and is therefore a Mori domain. This is impossible, since a Mori Prüfer domain is known to be a Dedekind domain. (To verify this, let I be an ideal of a Mori Prüfer domain R. Then, since R is Mori there is a finitely generated ideal $J\subseteq I$ with $I_v=J_v$. Since R is Prüfer, J is invertible, hence divisorial, and we have that $I=J$ is invertible.) □

For a Mori domain D, denote by $\mathscr{I}$ the set of v-invertible maximal divisorial ideals of D and by $\mathscr{S}$ the set of non-v-invertible maximal divisorial ideals of D. Recall from Barucci and Crabelli [6] that a Mori domain D has a canonical decomposition $D = D_1 \cap D_2$, where $D_1 = \bigcap_{P \in \mathscr{I}} D_P$ is a Krull domain (the "Krull part" of D) and $D_2 = \bigcap_{Q \in \mathscr{S}} D_Q$ is a strongly Mori domain (the "strongly Mori" part of D). (A strongly Mori domain is one in which each maximal divisorial ideal fails to be v-invertible.) Propositions 2.4 and 2.6 together imply that if we wish Int(D) to be Mori and nontrivial (Int(D) $\neq D[X]$), then every v-invertible maximal divisorial ideal, that is, every ideal in $\mathscr{I}$, must have infinite residue field, while some maximal divisorial ideal of D, necessarily in $\mathscr{S}$, must have finite residue field. This is impossible if D is a Krull domain, since in that case $\mathscr{S}$ is empty (and D reduces to its Krull part). Moreover, if D is a Krull domain, then so is $D[X]$. In view of Corollary 2.5, we then have the following result.

Corollary 2.7. *The following statements are equivalent for a Krull domain D*:
(1) Int(D) *is Mori.*
(2) Int(D) *is Krull.*
(3) Int(D) $= D[X]$.
(4) *Each height-one prime ideal of D has infinite residue field.*

We end this section with some comments which (partially) motivate the construction in Section 2 below. We consider the decomposition=$D = D_1 \cap D_2$, where $D_1 = \bigcap_{P \in \mathscr{I}} D_P$ is the Krull part of D and $D_2 = \bigcap_{Q \in \mathscr{S}} D_Q$ is the strongly Mori part of D. Now for each $P \in \mathscr{I}$, $\mathrm{Int}(D_1) \subseteq (\mathrm{Int}(D_1))_P = \mathrm{Int}(D_P)$, the equality following from Proposition 2.1 and the fact that $(D_1)_P = D_P$. On the other hand, we have $\bigcap_{P \in \mathscr{I}} \mathrm{Int}(D_P) \subseteq \bigcap_{P \in \mathscr{I}} \mathrm{Int}(D_1, D_P) = \mathrm{Int}(D_1, \bigcap_{P \in \mathscr{I}} (D_P)) = \mathrm{Int}(D_1)$. It follows that $\mathrm{Int}(D_1) = \bigcap_{P \in \mathscr{I}} \mathrm{Int}(D_P) = \bigcap_{P \in \mathscr{I}} (\mathrm{Int}(D))_P$ by Proposition 2.1. Thus $\mathrm{Int}(D_1)$ is a generalized ring of fractions of Int(D). Similarly, $\mathrm{Int}(D_2) = \bigcap_{Q \in \mathscr{S}} (\mathrm{Int}(D))_Q$ is a generalized ring of fractions of Int(D). In particular, since generalized rings of fractions of Mori domains are Mori [22], and since locally finite intersections of Mori domains are Mori [23, I, Théorème 2], we have that Int(D) is Mori if and only if both $\mathrm{Int}(D_1)$ and $\mathrm{Int}(D_2)$ are Mori. Thus if Int(D) is Mori, then by Corollary 2.7, its Krull part $\mathrm{Int}(D_1)$ is just the polynomial ring $D_1[X]$, and the study of its strongly Mori part $\mathrm{Int}(D_2)$ would be greatly simplified by an understanding of the rings $\mathrm{Int}(D_Q)$, where Q is a non-v-invertible maximal divisorial ideal of D. Thus we are particularly interested in the situation where D is a local Mori domain whose maximal ideal is strongly divisorial (non-v-invertible).

3. A Mori domain of integer-valued polynomials

In [16] Gilmer et al. give an example of a domain D such that Int(D) is Noetherian and Int(D) $\neq D[X]$, but they show that this cannot happen if D is one dimensional. Our first result of this section is to show that, in fact, if D is a one-dimensional

Noetherian domain such that $\mathrm{Int}(D) \neq D[X]$, then $\mathrm{Int}(D)$ cannot even be Mori. Our work in the remainder of the section shows that it is nevertheless possible to have D one-dimensional *Mori* with $\mathrm{Int}(D) \neq D[X]$ and $\mathrm{Int}(D)$ Mori. This part of our work may be viewed as a generalization of the study of integer-valued polynomial rings over pseudo-valuation domains given by Cahen and Haouat [12].

Theorem 3.1. *If D is a one-dimensional Noetherian domain such that $\mathrm{Int}(D)$ is a Mori domain, then $\mathrm{Int}(D) = D[X]$.*

Proof. Assume, by way of contradiction, that $\mathrm{Int}(D) \neq D[X]$, and let P be the upper to zero

$$P = \{f \in \mathrm{Int}(D) \mid f(0) = 0\} = XK[X] \cap \mathrm{Int}(D).$$

The first step is to show that $P^{-1} \subseteq K[X]$. As $\mathrm{Int}(D)$ is not trivial, there is a maximal ideal M of D such that D/M is finite. Then, since $P^{-1} \subseteq (P\,\mathrm{Int}(D_M))^{-1}$, we may assume that (D, M) is a local one-dimensional Noetherian domain with finite residue field. There is an overring R of D such that R is finitely generated as a D-module and such that R is locally unibranched. In particular, there is a nonzero element $d \in D$ such that $dR \subseteq D$. Let $Q = XK[X] \cap \mathrm{Int}(R)$; then Q is an upper to zero in $\mathrm{Int}(R)$, and $dQ \subseteq P$. Let $\varphi \in P^{-1}$. Since $X \in P$, we have $\varphi = f/X$ with $f \in K[X]$. We must show that, in fact, X divides f in $K[X]$, or, equivalently, that $f(0) = 0$. This just requires a modification of the proof of Cahen and Chabert [11, VIII.5.14]: Let $\{a_n\}$ be a sequence of nonzero elements of D such that $a_n \in M^n$, and let M' be a maximal ideal of R above M. Since R is locally unibranched, the integral closure of the localization $R_{M'}$ is the ring of a discrete rank-one valuation v. We clearly have $v(a_n) \geq n$. Since a_n is not a root of X, and since R is locally unibranched, the upper to zero Q is not contained in the maximal ideal $\mathscr{M}_{M',a_n} = \{g \in \mathrm{Int}(R) \mid v(a_n) > 0\}$ [11, Proposition V.3.3]. Consequently, there is a sequence $\{h_n\}$ of polynomials in $K[X]$ such that $Xh_n \in Q$ and $v(a_n h_n(a_n)) = 0$. We then have $dXh_n \in P$, so that $\varphi dXh_n = dfh_n \in \mathrm{Int}(D)$. It follows that $v(df(a_n)h_n(a_n)) \geq 0$, and hence $v(df(a_n)) \geq v(a_n) \geq n$ for each n. Thus $v(f(a_n))$ tends to infinity, and we have $f(0) = 0$. Therefore, $P^{-1} \subseteq K[X]$, as claimed.

Now, since $\mathrm{Int}(D)$ is a Mori domain and P has height one, P must be divisorial. Furthermore, we may write $P^{-1} = (\varphi_1, \ldots, \varphi_n)_v$, where $\varphi_i \in P^{-1}$ for $i = 1, \ldots, n$. Since each $\varphi_i \in K[X]$, there is an element $c \in D$ with $c\varphi_i \in \mathrm{Int}(D)$ for each i. Hence $cP^{-1} = c(\varphi_1, \ldots, \varphi_n)_v \subseteq \mathrm{Int}(D)$, and we have $c \in P_v = P$, which contradicts the fact that $P \cap D = (0)$. □

Throughout the rest of this section, we use the following notation. Let (B, M) be a (not necessarily Noetherian) local domain with quotient field K and with *infinite* residue field, let $I = aB$ be a proper nonzero principal ideal of B, let $\phi : B \to B/I$ the canonical map, and assume that B/I contains a *finite* field k. Then let $D = \phi^{-1}(k)$;

that is, let D be defined by the following pullback diagram:

$$\begin{array}{ccc} D & \longrightarrow & k \\ \downarrow & & \downarrow \\ B & \xrightarrow{\phi} & B/I. \end{array}$$

We denote by $d_0=0,\ d_1,\dots,d_{q-1}$ a complete set of coset representatives of D modulo I, and set

$$\varphi = \frac{\prod_{i=0}^{q-1}(X-d_i)}{a}.$$

Lemma 3.2. *With the notation above*, $\mathrm{Int}(D,B)=B[X][\varphi]$.

Proof. It is clear that $\varphi\in\mathrm{Int}(D,B)$. Conversely, if $f\in\mathrm{Int}(D,B)$, we shall prove that $f\in B[X][\varphi]$ by induction on $\deg f$. If $\deg f<q$, then, since $f(d_i)\in B$ for $i=0,\dots,q-1$, the usual Vandermonde determinant argument [11, Proposition I.3.1] shows that $f\in B[X]$. Otherwise, write $f=\varphi g+h$ with $g,h\in K[X]$ and $\deg h<q$ or $h=0$. Note that $h(d_i)=f(d_i)$ for each i and hence (as above) $h\in B[X]$. Now let v be a unit of B. Then for each i, $f(d_i+av)=\varphi(d_i+av)g(d_i+av)+h(d_i+av)$, and $\varphi(d_i+av)=v\prod_{j\neq i}(d_i-d_j+av)$ is a unit of B. It follows that $g(d_i+av)\in B$. The polynomial $g_i(X)=g(d_i+aX)$ is then such that $g_i(B^*)\subseteq B$ (where B^* is the set of units of B), and therefore, since B/M is infinite, [11, Proposition IV.1.20] implies that $g_i\in B[X]$. It follows that $g\in\mathrm{Int}(D,B)$, and we can conclude by induction since $\deg g<\deg f$. $\square$

We next wish to study the prime ideals of $\mathrm{Int}(D,B)$ above M and the prime ideals of $\mathrm{Int}(D)$ above I. Observe that the ideal $\mathrm{Int}(D,I)$ is shared by these two rings and that $\mathrm{Int}(D,I)$ is principal in $\mathrm{Int}(D,B)$ (generated by a). For each $x\in D$, let $M_x=\{f\in\mathrm{Int}(D,B)|=f(x)\in M\}$. It is known that M_x is a maximal ideal of $\mathrm{Int}(D,B)$ above M [11, Lemma V.1.3]. However, there are also some primes above M which are not maximal:

Lemma 3.3. *For each* $i=0,\dots,q-1$, *let* $P_i=\{f\in\mathrm{Int}(D,B)\mid f(d_i+I)\subseteq M)\}$. *Then*

(1) P_i *is a prime ideal of* $\mathrm{Int}(D,B)$ *above* M *which is not maximal,*

(2) $\mathrm{Int}(D,M)=\bigcap_{i=0}^{q-1}P_i$,

(3) $\{P_i\}_{i=0}^{q-1}$ *is precisely the set of primes of* $\mathrm{Int}(D,B)$ *which are minimal over* $\mathrm{Int}(D,M)$,

(4) $P_i=(M,X-d_i)\mathrm{Int}(D,B)$,

Proof. Let $\Phi_i : K[X]\mapsto K[X]$ be the morphism defined by $\Phi_i(f)=f(d_i+aX)$. Note that $\Phi_i(\varphi)\in B[X]$ and that $\Phi_i(B[X])\subseteq B[X]$. Hence by Lemma 3.2, Φ_i induces a morphism from $\mathrm{Int}(D,B)$ to $B[X]$. Now consider $f\in\mathrm{Int}(D,B)$. Reducing each coefficient modulo M produces a polynomial $\overline{\Phi_i(f)}$ in $(B/M)[X]$. Since B/M is infinite, this polynomial vanishes for all $\bar{x}\in B/M$ if and only if it is the zero polynomial; that is, $f(d_i + aX) \in M[X]$ if and only if $f(d_i + I) \subseteq M$. Therefore, $P_i = \Phi_i^{-1}(M[X])$. This proves that P_i is prime. Note that for each $x \in d_i + I$, we have $P_i \subseteq M_x$. On the other hand, $\varphi \in M_{d_i}$ (since $\varphi(d_i) = 0$), but $\varphi \notin P_i$ (since $\varphi(d_i + av)$ is a unit of B for each unit v of B). Hence P_i is not maximal. This proves (1). Statement (2) is clear, and (3) follows immediately from (2). To prove (4), let $g \in P_i$, and use Lemma 3.2 to write $g = g_0 + g_1\varphi + \cdots + g_r\varphi^r$, with $g_j \in B[X]$ for $j = 0, \ldots, r$. By assumption, $g(d_i) \in M$. Since $\varphi(d_i) = 0$, this implies that $g_0(d_i) \in M$, from which it follows that $g_0 \in (M, X - d_i)B[X]$. We also have $g_1\varphi + \cdots + g_r\varphi^r = g - g_0 \in P_i$. Since $\varphi \notin P_i$, we have $g_1 + \cdots + g_r\varphi^{r-1} \in P_i$. Continuing to argue in this manner, we can show that $g_j \in (M, X - d_i)$ for each j, as desired. □

We can now describe the primes of $\mathrm{Int}(D)$ above I:

Proposition 3.4. *For each* $i = 0, \ldots, q - 1$, *let* $\mathscr{M}_i = \{f \in \mathrm{Int}(D) \mid f(d_i) \in I)\}$. *Then*

(1) $\mathscr{M}_i$ *is a maximal ideal of* $\mathrm{Int}(D)$ *above* I *having residue field* k,
(2) $P_i \cap \mathrm{Int}(D) = \mathscr{M}_i$ (*where* P_i *is as defined in Lemma* 3.3),
(3) $\mathrm{Int}(D, I) = \bigcap_{i=0}^{q-1} \mathscr{M}_i$, *and*
(4) $\{\mathscr{M}_i\}_{i=0}^{q-1}$ *is precisely the set of prime ideals of* $\mathrm{Int}(D)$ *above* I.

Proof. Statement (1) is immediate (as noted in [11, Lemma V.1.3]). For (2), it is clear that $P_i \cap \mathrm{Int}(D) \subseteq \mathscr{M}_i$. Now observe that $I^2 = a^2B = aI \subseteq aD$, so that I is pseudo-principal. Hence by Cahen and Chabert [11, Proposition V.1.11], each prime ideal of $\mathrm{Int}(D)$ above I is maximal and contains $\mathrm{Int}(D, I)$. It follows that $P_i \cap \mathrm{Int}(D) = \mathscr{M}_i$. Hence, since $\mathrm{Int}(D, I) = \mathrm{Int}(D, M) \cap \mathrm{Int}(D)$, we have, using Lemma 3.3, that $\mathrm{Int}(D, I) = (\bigcap_{i=0}^{q-1} P_i) \cap \mathrm{Int}(D) = \bigcap_{i=0}^{q-1} \mathscr{M}_i$. This proves (3). Statement (4) follows easily. □

Remark 3.5. The preceding result generalizes [12, Lemme 3.1], which states (in the case where B is a rank one discrete valuation domain with maximal ideal I) that $\mathscr{M}_d = \mathscr{M}_{d'}$ if and only if $d \equiv d' \pmod I$.

We now wish to indicate conditions under which $\mathrm{Int}(D)$ will be a Mori domain. (Note that $\mathrm{Int}(D)$ is nontrivial since I is pseudo-principal [11, Propositions I.3.10 and I.3.12].) Of course, for this it is necessary to have D Mori by Proposition 2.6. In [24] Roitman extends the definition of Mori domain to domains without unit; thus an ideal of a domain R may be *Mori* (or not) regardless of whether R itself is a Mori domain. In [25], Roitman shows that a domain R is a Mori domain if (and only if) it contains a prime ideal P which is Mori and such that P is either maximal divisorial or satisfies $P_v = R$ [25, Theorem 4.14]. He also shows [25, Proposition 4.1] that any principal ideal of a Mori domain is Mori. Now in our construction, I is principal in B, and, as a maximal ideal of D, it is either maximal divisorial or satisfies $I_v = D$. Thus, in our construction, if B is a Mori domain, then so is D. We would be able to apply essentially the same argument to conclude that $\mathrm{Int}(D)$ is Mori if we knew that

$\mathrm{Int}(D,B)=B[X][\varphi]$ were Mori, but in [26] Roitman gives examples of Mori domains B for which even $B[X]$ is not Mori. However, if we assume that B is Noetherian, then all problems disappear:

Theorem 3.6. *If B is Noetherian, then* $\mathrm{Int}(D)$ *is Mori.*

Proof. To simplify the notation somewhat, we set $R=\mathrm{Int}(D)$, $T=\mathrm{Int}(D,B)$, and $J=\mathrm{Int}(D,I)$. The rings R and T share the ideal J, which implies that the following diagram is a pullback:

$$\begin{array}{ccc} R & \longrightarrow & R/J \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/J. \end{array}$$

Note that T is Noetherian by Lemma 3.2. We wish to localize this diagram with respect to $S_i=R\backslash\mathscr{M}_i$. Set $R_i=R_{S_i}$, $T_i=T_{S_i}$, and $J_i=J_{S_i}$. By Proposition 3.4, $J=\bigcap_{j=0}^{q-1}\mathscr{M}_j$. Hence $J_i=\mathscr{M}_iR_i$, and we have that R_i/J_i is a field isomorphic to k, again by Proposition 3.4. This produces a new pullback diagram:

$$\begin{array}{ccc} R_i & \longrightarrow & k \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & T_i/J_i. \end{array}$$

Since T_i is Noetherian, and J_i is principal in T_i and maximal in R_i, we can appeal to Roitman [25, Theorem 4.14] to conclude that R_i is a Mori domain. Moreover, it is not difficult to verify that $R=\left(\bigcap_{i=0}^{q-1}R_i\right)\cap T$. Therefore, since a finite intersection of Mori domains is Mori [23, I, Théorème 2], we conclude that R is a Mori domain. $\square$

Observe that, in particular, if B is a rank one discrete valuation domain, then D is a one-dimensional Mori domain for which $\mathrm{Int}(D)$ is Mori and nontrivial. We close this section with a somewhat more complicated construction.

Example 3.7. Let A be a Krull domain, and assume that $A=\bigcap_{\alpha\in\mathscr{A}}V_\alpha$, where each V_α is a rank one discrete valuation domain with infinite residue field. Let $\mathscr{B}$ be a finite subset of $\mathscr{A}$. For $\alpha\in\mathscr{B}$, let D_α be a Mori pullback as constructed above; and set $D_\alpha=V_\alpha$ for $\alpha\notin\mathscr{B}$. Then $R=\cap D_\alpha$, being a locally finite intersection of Mori domains, is a Mori domain by Raillard [23, I, Théorème 2]. Denote by M_α the maximal ideal of D_α, and let $P_\alpha=M_\alpha\cap R$. *Provided that the P_α are incomparable*, then [20, Corollary 8] implies that $D_{P_\alpha}=D_\alpha$ for each α. (Note: The P_α may be incomparable or not — see [6, Examples 4.6(a) and 4.6(b)]). It can then be shown that $\{P_\alpha\}_{\alpha\in\mathscr{A}}$ is precisely the set of maximal divisorial ideals of R, with $\{P_\alpha\}_{\alpha\in\mathscr{B}}$ being the set of strongly divisorial maximal divisorial ideals. Thus R is a Mori domain of t-dimension 1, and the canonical decomposition of R (see the discussion at the end of the first section) is $R=\left(\bigcap_{\alpha\in\mathscr{B}}D_\alpha\right)\cap T$, with $\bigcap_{\alpha\in\mathscr{B}}D_\alpha$ being strongly Mori and $T=\bigcap_{\alpha\in\mathscr{A}\backslash\mathscr{B}}D_\alpha$ being a

Krull domain. Since the D_α are localizations of R, we have $\mathrm{Int}(R) = (\bigcap_{\alpha\in\mathscr{B}} \mathrm{Int}(D_\alpha)) \cap \mathrm{Int}(T) = (\bigcap_{\alpha\in\mathscr{B}} \mathrm{Int}(D_\alpha)) \cap T[X]$. It follows that $\mathrm{Int}(R)$ is Mori [23, I, Théorème 2]

4. On the class group of Int(*D*)

We return to the general situation: D denotes a domain with quotient field K. We begin with a technical result on the behavior of the v- and t-operations in the extension $D \subseteq \mathrm{Int}(D)$.

Lemma 4.1. *Let I be a nonzero fractional ideal of a domain D. Then*
(1) $(I\,\mathrm{Int}(D))^{-1} = \mathrm{Int}(D, I^{-1}) = (\mathrm{Int}(D,I))^{-1}$,
(2) $(I\,\mathrm{Int}(D))_v = (I_v\,\mathrm{Int}(D))_v = (\mathrm{Int}(D,I))_v = \mathrm{Int}(D,I_v)$, *and*
(3) $(I\,\mathrm{Int}(D))_t = (I_t\,\mathrm{Int}(D))_t$.

Proof. If I and J are two fractional ideals of D, we clearly have $\mathrm{Int}(D,I)\mathrm{Int}(D,J) \subseteq \mathrm{Int}(D,IJ)$. It follows that $\mathrm{Int}(D,I^{-1}) \subseteq (\mathrm{Int}(D,I))^{-1} \subseteq (I\,\mathrm{Int}(D))^{-1}$ (the last containment, since $I\,\mathrm{Int}(D) \subseteq \mathrm{Int}(D,I)$). Conversely, let $f \in (I\,\mathrm{Int}(D))^{-1}$. For each $i \in I$, we have $i \in I\,\mathrm{Int}(D)$, and hence $if \in \mathrm{Int}(D)$. Therefore, for each $d \in D$, $if(d) \in D$. It follows that $f \in \mathrm{Int}(D,I^{-1})$. This proves (1).

For (2), the containments $I\,\mathrm{Int}(D) \subseteq I_v\,\mathrm{Int}(D) \subseteq \mathrm{Int}(D,I_v)$ and $I\,\mathrm{Int}(D) \subseteq \mathrm{Int}(D,I) \subseteq \mathrm{Int}(D,I_v)$ are clear. It follows from (1) that all these ideals have the same inverse, hence the same v-closure (being true for the extremes in each containment). Finally, $\mathrm{Int}(D,I_v)$ is divisorial (since it is the inverse of $\mathrm{Int}(D,I^{-1})$). This proves (2).

Statement (3) follows from (2) and [8, Proposition 2.1]. □

It is well known that divisorial ideals of D extend to divisorial ideals in $D[X]$. According to Lemma 4.1, however, this would not be the case in the extension $D \subseteq \mathrm{Int}(D)$ if D contains a divisorial ideal I for which $I\,\mathrm{Int}(D) \neq \mathrm{Int}(D,I)$. We next present such an example.

Example 4.2. Let V denote a valuation domain of the form $F+M$, where F is a field of characteristic 2 and M is the maximal ideal of V. Then let $D = k + M$, where k is the field with two elements. To have M divisorial in D, we need only have that D is a proper pullback, that is, that F properly contains k. If we specify that M is not principal in V, then by Cahen and Haouat [12, Proposition 2.2], $\mathrm{Int}(D) \subseteq V[X]$, so that $M\,\mathrm{Int}(D) \subseteq MV[X]$. Then, since $X^2 - X \in \mathrm{Int}(D,M) \setminus MV[X]$, we have $M\,\mathrm{Int}(D) \neq \mathrm{Int}(D,M)$.

We next wish to study class groups. We recall some notation and definitions. Denote the set of t-ideals of D by $t(D)$; this is a monoid under the t-product: $I \times J = (IJ)_t$. The ideal J is said to be *t-invertible* if $(JJ^{-1})_t = D$, and we denote the group of t-invertible t-ideals of D by $\mathscr{T}(D)$. We shall also denote the set of maximal t-ideals of D by t-max(D). The (t-)class group of D is then defined to be the quotient group $\mathrm{Cl}(D) = \mathscr{T}(D)/\mathscr{P}(D)$, where $\mathscr{P}(D)$ is the group of principal ideals of D. For a Krull domain, this is just the ordinary class group. This group has been studied in many papers [1–3,5,8–10,13–15]. By Lemma 4.1, the extension $D \subseteq \mathrm{Int}(D)$ is, in the terminology of Barucci [8] et al, *compatible* for the t-operation; that is, we have $I_t \subseteq (I\,\mathrm{Int}(D))_t$ for each nonzero ideal I of D. Hence by [8, Proposition 2.1], the map $\eta : t(D) \to t(\mathrm{Int}(D))$, given by $\eta(I) = (I\,\mathrm{Int}(D))_t$, is a (monoid) homomorphism, and it induces group homomorphisms $\theta : \mathscr{T}(D) \to \mathscr{T}(\mathrm{Int}(D))$ and $i : \mathrm{Cl}(D) \to \mathrm{Cl}(\mathrm{Int}(D))$ (the latter given by $i([I]) = [(I\,\mathrm{Int}(D))_t]$). Moreover, since $\mathrm{Int}(D) \cap K = D$ and $f\,\mathrm{Int}(D) \cap D = (0)$ for each $f \in \mathrm{Int}(D) \setminus D$, the maps η, θ, and i are injective [8, Lemma 1.3 and Theorem 1.4]. For convenience we state this formally.

Proposition 4.3. *Let D be a domain with quotient field K, and let the maps η, θ, and i be defined as above. Then all three maps are injective homomorphisms.*

Recall that the Picard group of D is defined by $\mathrm{Pic}(D) = \mathscr{I}(D)/\mathscr{P}(D)$, where $\mathscr{I}(D)$ is the group of invertible ideals of D. In [11, Proposition VIII.1.6] it is shown that one always has an exact sequence of the following type:

$$0 \longrightarrow \mathrm{Pic}(D) \xrightarrow{\ i\ } \mathrm{Pic}(\mathrm{Int}(D)) \xrightarrow{\ p\ } \prod_{M \in \mathrm{Max}(D)} \mathrm{Pic}((\mathrm{Int}(D))_M).$$

We shall show that a similar result holds for the class group. First, note that for each multiplicative subset S of D, we have a homomorphism $\mathscr{T}(D) \to \mathscr{T}(D_S)$, given by $I \mapsto (ID_S)_t$ [4, Theorem 2.2]. In fact, it is easy to show that $(ID_S)_t = (ID_S)$, since I is t-invertible. Again by Anderson et al. [4, Theorem 2.2], this induces a homomorphism $\mathrm{Cl}(D) \to \mathrm{Cl}(D_S)$, given by $[I] \mapsto [ID_S]$. In particular, we have, for each maximal t-ideal M of D, a homomorphism $p_M : \mathrm{Cl}(\mathrm{Int}(D)) \to \mathrm{Cl}((\mathrm{Int}(D))_M)$. This induces a homomorphism $p : \mathrm{Cl}(\mathrm{Int}(D)) \to \prod_{M \in t\text{-}\max(D)} \mathrm{Cl}((\mathrm{Int}(D))_M)$.

In what follows, we shall call an ideal of $\mathrm{Int}(D)$ *unitary* if it contains a nonzero element of D.

Proposition 4.4. *Let D be a domain with quotient field K. Then the following sequence is exact*:

$$0 \longrightarrow \mathrm{Cl}(D) \xrightarrow{\ i\ } \mathrm{Cl}(\mathrm{Int}(D)) \xrightarrow{\ p\ } \prod_{M \in t\text{-}\max(D)} \mathrm{Cl}((\mathrm{Int}(D))_M).$$

Proof. The map i is injective by Proposition 4.3. Let $[A] \in \mathrm{Im}(i)$. Then $[A] = [(I\,\mathrm{Int}(D))_t]$ for some t-invertible t-ideal I of D. Let $M \in t\text{-}\max(D)$. Since I is t-invertible, ID_M is principal, and it follows that $I\,(\mathrm{Int}(D))_M$ is principal. Hence $[A] \in \ker(p)$. Now let $[B] \in \ker(p)$. Since B is t-invertible, there is a finitely generated ideal C of $\mathrm{Int}(D)$ with $B = C_v$. By [11, Lemma VIII.1.2] there is an element $\phi \in K(X)$ such that ϕC is a unitary integral ideal of $\mathrm{Int}(D)$. Hence ϕB is also a unitary integral ideal of $\mathrm{Int}(D)$. Thus we may change notation and assume that B itself is a unitary integral ideal of $\mathrm{Int}(D)$. Let $I = B \cap D$; then I is a nonzero t-ideal of D

[8, Proposition 2.1]. We shall show that $B = I\,\mathrm{Int}(D)$. For this it suffices to show that $B(\mathrm{Int}(D))_{\mathscr{M}} = (I\,\mathrm{Int}(D))_t(\mathrm{Int}(D))_{\mathscr{M}}$ for each maximal t-ideal $\mathscr{M}$ of $\mathrm{Int}(D)$. If $\mathscr{M}$ is not unitary, then both ideals are equal to $(\mathrm{Int}(D))_{\mathscr{M}}$. Suppose that $\mathscr{M}$ is unitary. By Barucci [8, Proposition 2.1], $\mathscr{M} \cap D$ is a nonzero t-ideal of D. Let M be a maximal t-ideal of D containing $\mathscr{M} \cap D$. Then, since $[B] \in \ker(p)$, we have $B_M = g_M(\mathrm{Int}(D))_M$ for some $g_M \in B$. Since $I \neq (0)$, a degree argument shows that $g_M \in K$. Hence $g_M \in B \cap D = I$ (since $K \cap \mathrm{Int}(D) = D$). We then have $B(\mathrm{Int}(D))_M = g_M(\mathrm{Int}(D))_M = I(\mathrm{Int}(D))_M$. Since $D \setminus M \subseteq \mathrm{Int}(D) \setminus \mathscr{M}$, the desired equality follows from this.

It remains to show that I is t-invertible. Since B is t-invertible in $\mathrm{Int}(D)$, we have $(BB^{-1})_t = \mathrm{Int}(D)$. Choose a nonzero element b of $B \cap D = I$. Then bB^{-1} is an integral unitary t-invertible t-ideal of $\mathrm{Int}(D)$, and $[bB^{-1}] \in \ker(p)$. Hence by what was proved above, we can write $bB^{-1} = (J\,\mathrm{Int}(D))_t$ for some ideal J of D. Thus

$$(IJ\,\mathrm{Int}(D))_t = ((I\,\mathrm{Int}(D)) \cdot (J\mathrm{Int}(D)))_t = (BbB^{-1})_t = b\,\mathrm{Int}(D).$$

Hence by Proposition 4.3, we obtain $(IJ)_t = bD$. Thus I is t-invertible. $\square$

If in Proposition 4.4 we assume that D is a Mori domain, then $\mathrm{Cl}((\mathrm{Int}(D))_M) = \mathrm{Cl}(\mathrm{Int}(D_M))$ for each maximal t-ideal M of D. Then, since an ideal of a Mori domain is contained in only finitely many maximal t-ideals [17, Theorem 3.1], we have the following corollary.

Corollary 4.5. *Let D be a Mori domain. Then the following sequence is exact*:

$$0 \longrightarrow \mathrm{Cl}(D) \xrightarrow{i} \mathrm{Cl}(\mathrm{Int}(D)) \xrightarrow{p} \bigoplus_{M \in t\text{-}\max(D)} \mathrm{Cl}(\mathrm{Int}(D_M)).$$

We use the convention that a domain D has *t-dimension 1* if it is not a field and each maximal t-ideal has height 1.

Theorem 4.6. *Let D be a Mori domain of t-dimension* 1. *Then we have the following short exact sequence*:

$$0 \longrightarrow \mathrm{Cl}(D) \xrightarrow{i} \mathrm{Cl}(\mathrm{Int}(D)) \xrightarrow{p} \bigoplus_{M \in t\text{-}\max(D)} \mathrm{Cl}(\mathrm{Int}(D_M)) \longrightarrow 0$$

Proof. We adapt the proofs of Cahen and Chabert [11, Lemmas VIII.1.7 and VIII.1.8]. By Corollary 4.5 we need only demonstrate that p is surjective. We begin by working with a single maximal t-ideal M of D. Let A be an integral unitary t-invertible t-ideal of $\mathrm{Int}(D_M)$. Then A is v-finite, and since $\mathrm{Int}(D_M)$ is a localization of $\mathrm{Int}(D)$, we may write $A = (f_1, \ldots, f_s)_v$ with $f_i \in \mathrm{Int}(D)$ for each i. Also, $A \cap \mathrm{Int}(D)$ is a t-ideal of $\mathrm{Int}(D)$, whence $A \cap D$ is a t-ideal of D. Since D is Mori, we can write $A \cap D = (d_1, \ldots, d_m)_v$ with each $d_j \in D$. Let $B = (f_1, \ldots, f_s, d_1, \ldots, d_m)_v$, a t-ideal of $\mathrm{Int}(D)$. Note that $(B\,\mathrm{Int}(D_M))_t = A$. We also have $B \cap D = A \cap D$. Since D has t-dimension 1, it is easy to see that M is the only maximal t-ideal of D which contains $B \cap D$. Thus if N is a maximal t-ideal of D different from M, we have $B\,\mathrm{Int}(D_N) = \mathrm{Int}(D_N)$.

Next, we wish to see that B is t-invertible. Since B has finite type, it suffices to show that B is t-locally principal. (That this is sufficient is well known. A specific reference for a more general result is [19, Proposition 2.6].) To this end, let $\mathcal{M}$ be a maximal t-ideal of $\text{Int}(D)$ containing B. Then $\mathcal{M} \cap D$ is a prime t-ideal of D containing $B \cap D$, and we have $\mathcal{M} \cap D = M$. We claim that $\mathcal{M}\,\text{Int}(D_M)$ is a maximal t-ideal of $\text{Int}(D_M)$. To verify this, first note that, since D is a Mori domain, there is a finitely generated ideal J of R with $J_v = M$, and hence also with $(JD_M)_v = MD_M$. We now split the proof of the claim into two cases.

The first case is that M has infinite residue field. In this case, [11, Proposition I.3.4] guarantees that $\text{Int}(D) \subseteq D_M[X] = \text{Int}(D_M)$. We shall show that $\mathcal{M}\,\text{Int}(D_M) = MD_M[X]$, which is a maximal t-ideal of $D_M[X]$ by Houston and Zafrullah [18, Proposition 2.1]. For this it suffices to show that $\mathcal{M} \subseteq MD_M[X] \cap \text{Int}(D)$. Suppose, on the contrary, that there is an element $f \in \mathcal{M}$ with $f \notin MD_M[X]$. By Lemma 4.1, $((J,f)D_M))_v$ contains $(JD_M)_v = MD_M$. This, coupled with the fact that $MD_M[X]$ is a maximal t-ideal of $D_M[X]$, implies that $((J,f)\text{Int}(D))^{-1} \subseteq ((J,f)D_M[X])^{-1} = D_M[X]$. Hence if $g \in ((J,f)\text{Int}(D))^{-1}$, then there is an element $s \in D \setminus M$ with $sg \in D[X] \subseteq \text{Int}(D)$. Thus $g(J,s) \subseteq \text{Int}(D)$. However, $(J,s)_v = D$, and since Lemma 4.1 then implies that $((J,s)\text{Int}(D))_v = \text{Int}(D)$, we have $g \in \text{Int}(D)$. This shows that $((J,f)\text{Int}(D))^{-1} = \text{Int}(D)$ and hence that $((J,f)\text{Int}(D))_v = \text{Int}(D)$, which contradicts the fact that $\mathcal{M}$ is a t-ideal of $\text{Int}(D)$. It follows that $\mathcal{M}\,\text{Int}(D_M) = MD_M[X]$, as desired.

The second case is that M has finite residue field. Since D_M is one-dimensional, we have that if a is a nonzero element of MD_M, then $(JD_M)^n \subseteq aD_M$ for some $n \geq 1$. Thus $(MD_M)^n = (J_vD_M)^n \subseteq ((JD_M)^n)_v \subseteq aD_M$, so that MD_M is pseudo-principal. Since $\mathcal{M}\,\text{Int}(D_M)$ is a prime ideal of $\text{Int}(D_M)$ above MD_M, it follows from Cahen and Chabert [11, Proposition V.1.11] that $\mathcal{M}\,\text{Int}(D_M)$ is a maximal ideal of $\text{Int}(D_M)$ which is minimal over $a = \text{Int}(D_M)$. Hence $\mathcal{M}\,\text{Int}(D_M)$ is a maximal t-ideal of $\text{Int}(D_M)$ in this case as well.

Now, since $D \setminus M \subseteq \text{Int}(D) \setminus \mathcal{M}$, we have $B(\text{Int}(D))_{\mathcal{M}} = (B\,\text{Int}(D_M))_{\mathcal{M}\,\text{Int}(D_M)}$. Since $(B\,\text{Int}(D_M))_t = A$ A is t-invertible, and $\mathcal{M}\,\text{Int}(D_M)$ is a maximal t-ideal, $(B\,\text{Int}(D_M))_{\mathcal{M}\,\text{Int}(D_M)}$ is principal. Thus B is t-invertible.

It remains to globalize. An element ξ of $\bigoplus_{M \in t\text{-max}(D)} \text{Cl}((\text{Int}(D_M))$ has only finitely many nontrivial components $A_1, \ldots, A_n$, corresponding to the maximal t-ideals $M_1, \ldots, M_n$ of D, and we may assume that each A_i is an integral unitary t-invertible t-ideal of $\text{Int}(D_{M_i})$. For each i, produce a corresponding ideal B_i as above. It is then easy to see that $p\big(\big[(\prod_{i=1}^n B_i)_t\big]\big) = \xi$. Hence p is surjective, and the proof is complete. □

Let D be the domain of Theorem 3.6. The nonzero prime ideals of $\text{Int}(D)$ are of two types. The primes above I are given by Proposition 3.4, and these are all maximal. The other primes must contract to (0) in D. These primes, called *uppers to zero*, are contracted from (and are in one-to-one correspondence with) the nonzero primes of $K[X]$. Thus each nonzero prime ideal is a t-ideal: the uppers to zero because they have height one, and the $\mathcal{M}_i$ because they are minimal over the ideal $a\,\text{Int}(D)$. The $\mathcal{M}_i$ are maximal t-ideals (since they are maximal), and, since they have height two (note that $\mathscr{M}_i \supset (X - d_i)K[X] \cap \mathrm{Int}(D)$), they cannot be v-invertible [6, Theorem 2.5]. Since localizations at uppers to zero are rank one discrete valuation domains, an upper to zero which is maximal is v-invertible by [6, Theorem 2.5]. On the other hand, since a nonmaximal upper to zero is contained in some $\mathscr{M}_i$ and is therefore not a maximal t-ideal, [18, Proposition 1.3] implies that it cannot be v-invertible. Hence the canonical decomposition of Barucci and Gabelli [6] is $\mathrm{Int}(D) = \left(\bigcap_{i=0}^{q-1} (\mathrm{Int}(D))_{\mathscr{M}_i}\right) \cap T$, where T is an intersection of localizations at (maximal) uppers to zero. Since $\left(\bigcap_{i=0}^{q-1} (\mathrm{Int}(D))\right)_{\mathscr{M}_i}$ is semilocal, [7, Corollary 2.12] shows that its class group is zero, and [7, Remark 2.7(2)] then shows that $\mathrm{Cl}(\mathrm{Int}(D)) \approx \mathscr{D}(T)/H$, where $\mathscr{D}(T)$ is the group of divisor classes of T and H consists of the divisorial (classes of) ideals of T whose contractions to $\mathrm{Int}(D)$ are principal. It follows, again using [7, Remark 2.7(2)], that $\mathrm{Cl}(\mathrm{Int}(D))$ is generated by the classes of the uppers to zero which are maximal t-ideals.

Finally, let $R = \bigcap D_\alpha$ be as in Example 3.7. Then R has t-dimension 1, and $\mathrm{Int}(R)$ is a Mori domain. It is not difficult to check that the non-t-invertible maximal t-ideals of $\mathrm{Int}(R)$ are precisely the contractions to $\mathrm{Int}(R)$ of the maximal ideals of the $\mathrm{Int}(D_\alpha)$ for $\alpha \in \mathscr{B}$. Hence $\mathrm{Int}(R)$ has only finitely many non-t-invertible maximal t-ideals, and, as before, we can conclude that $\mathrm{Cl}(\mathrm{Int}(D))$ is generated by the classes of the t-invertible maximal t-ideals. Now suppose that we arrange to have $\mathrm{Cl}(R) = 0$. (For example, let $\mathscr{A}$ be finite and take $\mathscr{B} = \mathscr{A}$.) Then by Theorem 4.6 have $\mathrm{Cl}(\mathrm{Int}(R)) = \left(\bigoplus_{\alpha \in \mathscr{B}} \mathrm{Cl}(\mathrm{Int}(D_\alpha))\right) \oplus \left(\bigoplus_{\alpha \in \mathscr{A}\setminus\mathscr{B}} \mathrm{Cl}(\mathrm{Int}(D_\alpha))\right)$. For $\alpha \in \mathscr{A} \setminus \mathscr{B}$, we have $\mathrm{Cl}(\mathrm{Int}(D_\alpha)) = \mathrm{Cl}(D_\alpha[X]) = 0$ since D_α is a rank one discrete valuation domain. Hence we obtain $\mathrm{Cl}(\mathrm{Int}(R)) \approx \bigoplus_{\alpha \in \mathscr{B}} \mathrm{Cl}(\mathrm{Int}(D_\alpha))$.

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